General Equilibrium: Arbitrage and Information*

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Abstract

Adopting a non-probabilistic formulation of the Efficient Markets Hypothesis, this paper looks at its relationship to general equilibrium theory. Shannon’s entropy allows us to show that arbitrage-free prices maximise the economy-wide amount of information, thereby bringing the two concepts together.

Keywords: arbitrage, entropy, information, efficient markets

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1 Introduction

This paper addresses the relationship between efficient markets and general equilibrium theory (GET). For an overview of the Efficient Market Hypothesis (EMH) in a traditional probabilistic setting see Shiryaev [1999]. Given this formulation, illustrations using real financial data are straightforward. On the other hand, difficulties involved in defining ‘information’ rigorously made alternative studies more complicated. This, and the existing volume of financial literature, have obscured the fact that Hayek [1945] brought forward an idea fundamentally similar to the basics of EMH, which later became an implicit part of GET. See Shleifer [2000] for a persuasively critical recent exposition of the issues involved.

The main implication of GET – the existence of a positive market-clearing price vector $p^*$ – has diverted attention from the possibility of obtaining $p^*$ if the only piece of economy-wide information available to agents is the price vector $p$. In order to concentrate on this phenomenon, this paper suggests a more immediate analysis of prices and associated information. Shannon’s entropy shows clearly how arbitrage-free prices maximise a well-defined measure of the amount of information. This approach, whilst rigorous, is simpler than the probabilistic theory of financial markets and allows to introduce the topic without obscuring it by an excessive amount of supporting theory. For another alternative approach see Kihlstrom and Mirman [1975].

The structure is as follows. First, some concepts of a GET model are introduced and the implied informational structure is discussed. This is followed by formal derivations needed to obtain the central result. A discussion of some associated results concludes.
2 Arbitrage

Consider a set $N$ of agents with $|N| = N$, where $N = \text{const} < \infty$ ($|A|$ denotes the number of elements in a set $A$), here thought of as consumers and/or producers. Commodities are elements of a set $G$ with $|G| = G = \text{const} < \infty$. Being goods or services whose full physical descriptions, spatial and temporal locations are unambiguous and constitute common knowledge, their existence implies that no ‘lemons’ in the sense of Akerlof [1970] can be unknowingly traded. ‘Lemons’ are simply different commodities. With these differences known to all agents there can be no confusion and no associated complications. Commodities, therefore, constitute an important tool in describing the informational structure of an ideal market economy. Here information is said to be perfect if and only if commodities can be defined in an unambiguous, universally known and verifiable way.

Given that a commodity – effectively a forward contract – absorbs all temporal dynamics in a GET model, arbitrage in the present context is purely ‘spatial’ (inter-agent). It refers to the possibility of making abnormal profits from spatial informational deficiencies, e.g., by charging different trading partners different prices for the same commodity.

We assume that agents are free to trade with anybody they wish, so that ‘zero transaction costs’ are interpreted in a broad sense, implying the absence of physical and/or institutional barriers to trade. Furthermore, agents are unable to influence each others’ decision-making either because they are small relative to the market or because market power cannot be used due to institutional constraints. In effect all trades are done anonymously in a single market stall where everybody can observe prices and types of all commodities offered by everybody else. All commodities are known to have been checked by an independent and trusted third party to ensure that all characteristics correspond to established
Agents make independent decisions given utility functions \( \{u_n(\cdot)\}_{n=1}^N \) satisfying all the usual assumptions (see Debreu [1959]) and non-zero vectors of endowments, which together constitute private information, and a positive vector of prices, \( \mathbf{p} \), which is common knowledge. The main result of GET is the existence of a positive vector of prices which clears all markets. Here we concentrate on the existence of an economy-wide constant price vector \( \mathbf{p} \).

Let \( G^{[n]} \) be the set of commodities traded by the \( n \)th agent (see below), and let \( N^{[n]}_g, g \in G^{[n]} \), be the \textit{ex ante} set of trading partners of that agent in the market for the \( g \)th commodity. Clearly,

\[
\bigcup_{n=1}^N G^{[n]} = \mathcal{G}, \quad \text{and} \\
\bigcup_{n=1}^N \bigcup_{g \in G^{[n]}} N^{[n]}_g = \mathbb{N}.
\] (2.1)

Due to different sets of trading partners, buyers have to be explicitly distinguished from sellers. Let \( b^{[n]}_g \) and \( s^{[l]}_g \) denote, respectively, the sets of buyers (observed by sellers) and sellers (observed by buyers) that the \( n \)th and the \( l \)th agents face in the \( g \)th market. Perfect information and zero transaction costs imply

\[
b^{[n]}_g = b^{[k]}_g = b_g, \quad \text{and} \\
s^{[l]}_g = s^{[j]}_g = s_g.
\] (2.2)

In what follows only actions of buyers will be looked at, unless explicitly stated otherwise.

The optimisation process is best thought of as comprising two stages. Firstly, by the usual arguments (see Debreu [1959]) utility functions \( \{u_n(\cdot)\}_{n=1}^N \) define
the sets $\mathcal{G}^{[n]}$ for all $n \in \mathbb{N}$ as the outcome of individual wealth-constrained optimisation. Secondly, given $\mathcal{G}^{[n]}$, agents seek to obtain greatest possible quantities at lowest possible cost by establishing ‘lowest asking prices’

$$l \in s_{\mathcal{N}_g} : \quad p_{[n,l]}^{[g]} = \min_{j \in s_{\mathcal{N}_g}} \{ p_{[n,j]}^{[g]} \} \equiv p_{[n]}^{[g]} \quad \forall \ g \in \mathcal{G}^{[n]},$$

for all $n \in \mathcal{N}_g$, where $\mathcal{N}_g := \bigcup_{g \in \mathcal{G}} b_{\mathcal{N}_g}$ and $p_{[n,l]}^{[g]}$ is the price of the $g$th commodity quoted to $n$ by $l$. Perfect information implies

$$\min_{l \in s_{\mathcal{N}_g}} \{ p_{[n,l]}^{[g]} \} = \min_{j \in s_{\mathcal{N}_g}} \{ p_{[k,j]}^{[g]} \} \quad \forall \ n, k \in b_{\mathcal{N}_g}. \quad (2.3)$$

This, together with spatial arbitrage, implies that the qualitative event of a transaction ($n$ buys from $l$) conveys information, a fact that will be used in Section 3. Given $s_{\mathcal{N}_g}$, let $\pi_{[n,l]}^{[g]}$ be the ex ante probability that the $n$th agent will trade with the $l$th (denote this by “($n \leftrightarrow l$)-trade”). Unambiguously verifiable characteristics of all commodities, perfectly competitive free markets and wealth-constrained optimisation lead to

$$\pi_{[n,l]}^{[g]} = \pi_{[n]}^{[g]} \left( p_{[n,l]}^{[g]} \right).$$

Note that the probability of an ($n \leftrightarrow l$)-trade is a function of only the price of the $g$th good quoted for that sale. From (2.2), every buyer is able to observe the full set $s_{\mathcal{N}_g}$ and compare quoted prices. All commodity-defining characteristics are by definition constant throughout $s_{\mathcal{N}_g}$ and consequently cannot affect the choice of a particular seller. Furthermore, the adopted interpretation of zero transaction costs implies that no seller $l$ is more convenient for a buyer $n$ than any other seller $k$, so that ($n \leftrightarrow l$)-trade in the $g$th market and ($n \leftrightarrow k$)-trade for $k \in s_{\mathcal{N}_h}$ in a different market $h \neq g$ are independent from one another.
Given perfect information, supply and demand curves are well-behaved, and therefore a higher value of \( p_g^{[n,l]} \) does not signal higher quality of commodities supplied by the \( l \)th seller. In view of (2.2) and (2.3) the functions \( \pi_g^{[n]}(\cdot) \) can be written

\[
\pi_g^{[n]}(p_g^{[n,l]}) = 0 \quad (2.4)
\]

if \( p_g^{[n,l]} > p_g^{[n]} \) for any \( l \in sN_g \), and

\[
\pi_g^{[n]}(p_g^{[n,l]}) = \text{const}, \quad (2.5)
\]

for all \( l \in sN_g \) such that \( p_g^{[n,l]} = p_g^{[n]} \). These two expressions agree with classical results on competition in free markets under perfect information. The first of the two equalities above is simply the traditional kinked demand curve, whereas the second arises from the specific conditions postulated in this paper.

Following the above definition and discussion of probability functions \( \pi_g^{[n]}(\cdot) \) we can refine (2.5) even further:

\[
\pi_g^{[n]}(p_g^{[n]}) = \frac{1}{|sN_g|}, \quad (2.6)
\]

where the denominator on the right-hand side denotes the number of agents selling type-\( g \) goods at \( p_g^{[n]} \). Since the \( n \)th agent was chosen arbitrarily, the same results hold for all other buyers, and expressions (2.1), (2.2) together with (2.4) yield

\[
p_g^{[n,l]} = p_g^{[j,k]} = \text{const} \quad \forall n, j \in sN_g, \quad \forall l, k \in sN_g
\]

for all \( g \in S \). In other words, all agents observe identical prices for all commodities, which removes the possibility of arbitrage as defined in this paper.
3 Information

Let the usual definition of the strong form-EMH (due to Fama [1970]) be written as: arbitrage-free prices maximise information. Recall from Section 2 that the event of a transaction conveys information and note that with \( n \) possible realisations \( x_i \in x \) and \( n \) associated probabilities \( \pi_i = \pi(x = x_i) \), \( 1 \leq i \leq n \), Shannon’s amount of information (or ‘entropy’, see Shannon’s work reprinted in Shannon and Weaver [1964]) for an event is given by

\[
H(x) = -\sum_{i=1}^{n} \pi_i \cdot \ln(\pi_i), \quad 0 < \pi_i \leq 1, \quad \sum_{i=1}^{n} \pi_i = 1.
\]

Straightforward constrained optimisation shows that \( \pi_i = \pi = 1/n \) maximises \( H(x) \).

The amount of information \( H(n, g) \) associated with the \( n \)th agent acting in the market for the \( g \)th commodity is

\[
H(n, g) = -\sum_{l \in s, N_g} \pi_{n,l}^{n,l} \cdot \ln \left( \pi_{g}^{n,l} \right).
\]

(3.1)

It follows that \( H(n, g) \) is maximised by \( \pi_{g}^{n,l} = \pi_{g}^{n,k} \) for all \( k, l \in s, N_g \). Relying on the arguments preceding equation (2.4) and using the fact that the entropy of joint occurrence of independent events equals the sum of individual entropies of these events (Shannon and Weaver [1964, chap. 2]), the aggregate amount of information across all agents and all markets equals the sum of individual quantities equivalent to (3.1). Therefore, equations (2.4)-(2.6) and (3.1) together imply that with perfect information markets will be arbitrage-free, the amount of information will be maximised, and traditional general equilibrium analysis deals with markets that are efficient in this sense.

\[ \text{Note that in Shannon’s presentation ‘greater information’ amounts to ‘greater choice’ and thus to greater uncertainty concerning the ultimate outcome.} \]
These concepts can be used to analyse the relationship between market structure and information. Equations (2.2), (2.3), and (3.1) allow the calculation of the amount of information for the \( n \)th buyer in the \( g \)th market

\[
H_b(n, g) = \ln |s_{Ng}| \geq 0, \quad (3.2)
\]

and independence of decision-making allows to aggregate it across all buyers and all markets:

\[
H_b(g) = |b_{Ng}| \cdot \ln |s_{Ng}| \geq 0, \quad \text{and}
\]

\[
H_b = \sum_{g \in \mathcal{G}} |b_{Ng}| \cdot \ln |s_{Ng}| \geq 0. \quad (3.3)
\]

Some appealing conclusions can be drawn from the last expression. Note that

\[
\min_{g \in \mathcal{G}} \{|b_{Ng}|\} = \min_{g \in \mathcal{G}} \{|s_{Ng}|\} = 1,
\]

and (3.2)-(3.3) hold with equality if and only if \( |s_{Ng}| = 1 \). This means that although a monopoly ensures trivially simple purchasing decisions, the associated amount of information is lower than it would have been if numerous suppliers were competing in the \( g \)th market: monopolistic pricing does not allow aspects of individual firms to be revealed through competition. A monopolist is by definition the fittest on the market, although if competition was to take place this may prove no longer to be the case, thereby increasing the amount of information (and raising efficiency).

An approach from the sellers’ perspective produces qualitatively similar results, and an upper bound on the amount of information in an economy is

\[
\mathcal{H} = 2G \cdot N \ln(N - 1) \quad (3.4)
\]
which confirms the above reasoning. This corresponds to the artificial case where all agents are endowed with all commodities, have identical domains of their utility functions, and are equally willing to buy and sell any commodity that they possess.

4 Empirics

An interesting empirical application of the above theoretical constructs is possible. From (3.4) the amount of information $H$ increases linearly with ‘complexity’ – the number of intermediate goods used in production – in the specific sense of the empirical study by Blanchard and Kremer [1997] (referred to as ‘B&K’ below), and it increases logarithmically with a measure inversely related to their concept of ‘specificity’, which is the number of alternative suppliers of each intermediate good. By establishing this relationship – most notably in the discussion surrounding (3.2)-(3.3) – the present analysis provides a theoretical justification of the meanings assigned to ‘complexity’ and ‘specificity’.

The study by B&K presents the decline of output in transition economies as a result of hold-up problems (see Hart [1995]) arising from higher ‘specificity’ of centrally planned economies (CPEs), and omits a possible comparison of centrally planned and market economies within an informational framework. Yet this comparison can be incorporated by noting that since $sN_g$ in (3.2)-(3.3) corresponds to the set of suppliers of commodity $g$, ‘specificity’-induced problems faced by firms using this commodity as an input are proportional to $|sN_g|^{-1}$. Empirically this quantity is close to the nature of ‘specificity’, whereas ‘reported shortages’ used by B&K are closer to its effects. Recalling that $G^{[n]}$ is the set of commodities required by the $n$th agent makes $|G^{[n]}|$ a measure of ‘complexity’, which in the empirical section of their paper B&K define as “1 minus the Herfindahl index of input concentration for sector $i$”. Using the
above notation the Herfindahl index for sector $g$ can be defined as follows. Let

$$G_g^{[n]} \equiv \bigcup_{n \in \mathcal{N}_g} G^{[n]}$$

be the set of all goods used in the $g$th sector, and let $v_j^n$ denote the quantity of good $j$ used by $n \in \mathcal{N}_g$. Then

$$\sum_{j \in G_g^{[n]}} (\phi_{j,g})^2$$

is the Herfindahl index of input concentration for sector $g$, where

$$\phi_{j,g} \equiv \frac{\sum_{n \in \mathcal{N}_g} v_j^n}{\sum_{j \in G_g^{[n]}} \sum_{n \in \mathcal{N}_g} v_j^n}.$$

On a more disaggregated level consistent with previous section, taking (3.2) and aggregating it over $G_g^{[n]}$ we obtain $\sum_{g \in G^{[n]}} \ln |\mathcal{N}_g|$. This quantity brings together ‘complexity’ and ‘specificity’ for a single firm within the informational framework suggested in this paper and allows to conclude that the amount of information increases with ‘complexity’ and declines with ‘specificity’. Aggregating this quantity across all $n \in \mathcal{N}_g$ generates a sector-wide measure which brings together within the above framework the results of B&K concerning complexity and specificity presented in Tables I-VI in their cited paper. Furthermore, using as a postulate their result that CPEs are more ‘specific’ than market economies the above definitions allow us to conclude that for a CPE which is as ‘complex’ as some market economy, *ceteris paribus* the amount of information is smaller.
5 Conclusion

Using the informational assumptions of what was previously treated as largely distinct areas of economics, we established a link between the theory of financial markets and GET. We showed that an economy-wide constant price vector $p$ follows from the existence of an ‘ideal’ economy, and that economic inefficiencies reduce the economy-wide amount of information. This observation has plausible empirical applications. For example, the framework suggested in this paper can be used to suggest how the measures of ‘complexity’ and ‘specificity’ used by Blanchard and Kremer [1997] can be applied to empirical comparisons of economies with different sets of institutions using the amount of information from (3.4) as an evaluation criterion.
References


